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Instability of a two-dimensional compressible jet

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A linearized analysis of the two-dimensional double vortex sheet model of a jet shows that inviscid jet instabilities occur over a wide range of frequencies at all jet Mach numbers. No particular frequency for maximum growth rate exists unless finite shear layer thickness effects are considered. It is suggested that the model describes the essential characteristics of a real jet disturbed by long wavelength perturbations. The idea is advanced that the jet flow constitutes a broad band amplifier of high gain. Disturbances can grow rapidly to a size when non-linear effects bring about significant interaction with the mean flow. By seeding the jet with disturbances of a type that are highly amplified it is argued that gross features of the flow may be affected and that the jet may be rendered less noisy at high Mach number. It is argued that some of these ideas are supported by the observation that a supersonic jet diffuses at an unusually rapid rate when subject to the oscillatory condition known as 'screech'.

1. Introduction

The time-dependent behaviour of high-speed jets is of great interest in the study of jet noise. In this paper we study the possible role of instabilities that are driven by the velocity difference between the jet and the ambient fluid for arbitrary Mach number. The study is confined to a model of an infinite, two-dimensional, inviscid jet separated from the motionless ambient fluid by parallel plane vortex sheets. The model is, of course, chosen for analytical simplicity but it is intended to indicate well the qualitative features of real jet flows with unsteady disturbances of wavelength long compared to the shear layer thickness. Shear layers are known to have features independent of the velocity profile in this limit, a point demonstrated by the work of Lord Rayleigh (1945), Wille (1963), and Graham & Graham (1968). The shear layers of jets of engineering interest are turbulent and it is an observed fact that turbulence levels tend to fall with increasing Mach number (Lighthill 1954). Landahl (1967) has argued with some success that the unsteady properties of turbulent flow can be established by regarding the mean flow profile as excited by second-order effects to produce the first-order unsteady motions largely corresponding to the least stable modes of a steady inviscid flow with the same mean velocity profile. His theory is essentially linear and this present argument is in the same spirit. We postulate that for disturbances of long wavelength, linear modes of the mean velocity profile

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(independent of the profile) can be superposed on the turbulent motion, so that the steady flow model provides a proper simulation of the real jet.

The analytical scheme used in this paper is that developed by Miles (1958) to study the dynamics of compressible vortex sheets. We will examine in detail the double vortex sheet idealization of a two-dimensional jet. We find that the jet, unlike the isolated vortex sheet, is unstable to a wide range of disturbances for all Mach numbers. The main outcome of the study is the viewpoint that the jet can be regarded as a high gain broad band amplifier. The computation of the 'gain', or the amplification rates of small disturbances, constitutes the details of the work. However, it is the general structure of the unsteady flow that we are now proposing to be of relevance to the jet noise issue. We argue its relevance as follows. We view the jet flow as an amplifier because small disturbances grow exponentially as they travel downstream. Any feedback loop on such an amplifier is liable to transform the system into a narrow band oscillator. This is our view of the jet 'screech' phenomenon, an oscillatory condition common to non-ideally expanded cellular supersonic jet flows. The feedback loop is described by Powell (1953). A disturbance in the jet shear layer is convected downstream, and strikes a cell boundary to scatter intense sound by non-linear interaction. This sound propagates through the subsonic ambient fluid, interacts with the lip of the jet nozzle, and produces a new downstream travelling disturbance that continues the cycle. The essential element of the cycle, the gain in the strength of the downstream travelling disturbance that supplies the energy to overcome radiation and viscous losses, is, we consider, a property of the basic mean flow and is present over a broad band of frequencies. It is only the frequency of operation that is set by the feedback loop. The details of the flow field are computable from stability theory once the frequency of oscillation is known from Powell's argument. According to this view the jet could be *excited* into harmonic motion at non-screech frequencies if driven by an oscillatory upstream disturbance. The resulting motion could be just as violent as that experienced by a jet in screech. In particular the non-linear breakdown of the high amplitude shear layer perturbation can bring about the rate of jet diffusion observed in a screeching jet—which is very much in excess of the rate of normal turbulent jet diffusion. It is currently being argued that the broad band noise of a supersonic jet will be minimized when the diffusion rate of the jet is maximized, and it is to this aspect of the noise problem that we suggest this paper is relevant. Rapid jet spread is known to occur in screech as a result of amplified oscillatory motion. It is possible that it may be induced in non-screeching jets by forced oscillatory motion. The resulting motion would not inevitably be a powerful generator of discreet frequency sound. Though such sound would be present to some extent, we would not expect it to be of comparable magnitude to that of jet screech. The reason for this is that jet screech is dependent on acoustic feedback through the static environment. Any large amplitude oscillation must in that circumstance be associated with a large amplitude sound field. Indeed jet screech is a notoriously noisy phenomenon. In a forced jet the situation could be quite different since the jet motion would result from the internal seeding of an initial disturbance and a powerful discreet sound field is in no way an essential or inevitable consequence.

The details of flow régimes that may be excited start with a study of the possible linear motions of an ideal jet flow. Growth rates and mode shapes computed from this theory can form the starting point for a more quantitative description of the motions that we suggest are excitable in a real high Reynolds number jet flow. That study is described in this paper.

2. Inviscid jet instabilities

The inviscid instabilities of a jet will be studied by means of the following idealized problem illustrated in figure 1. An infinite, two-dimensional, inviscid jet with a constant velocity profile of arbitrary magnitude is separated from the motionless ambient fluid by infinite plane vortex sheets. Perturbed solutions of the form

$$\phi_j \sim \exp\left[i\alpha(x-\bar{c}t \pm \beta_j y)\right] \qquad (j = 1, 2) \tag{1}$$

are considered. The suffix '1' refers to the outer stagnant fluid and '2' to the jet region. Each of the two sets of solutions satisfies the appropriate wave equation,

$$(1-M_j^2)\,\phi_{jxx} + \phi_{jyy} - (1/a_j^2)\left[\phi_{jtt} + 2U_j \phi_{jxt}\right] = 0, \tag{2}$$

where $M_j = U_j/a_j$, U_j is the mean flow velocity and a_j is the speed of sound in each region. β_j is defined by

$$\beta_j^2 = (c_j - M_j)^2 - 1, \tag{3}$$

$$c_j = \bar{c}/a_j, \quad \text{and} \quad U_1 = 0, \quad U_2 = U, \quad M_2 = M.$$

If one arbitrarily chooses the sign of the square root of β_1^2 such that $\mathrm{Re}\,\beta_1 > 0$ then the plus sign in (1) is chosen in the region above the jet so that disturbances propagate away from the jet, and similarly the minus sign is chosen for the region below the jet. In region 2 within the jet there are two solutions, each using a different sign.

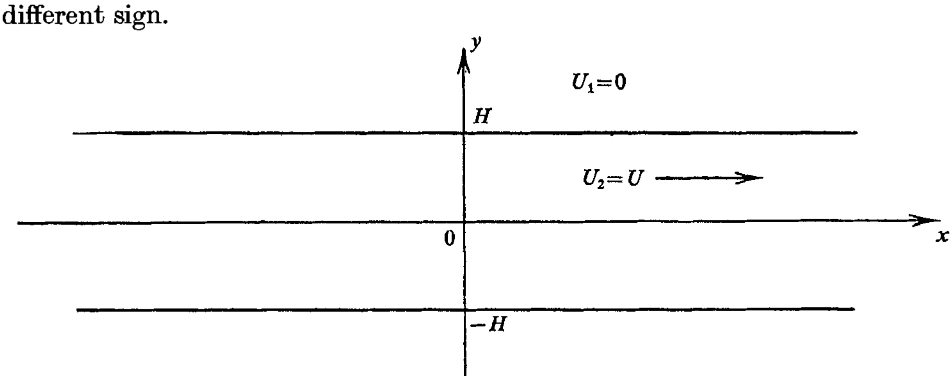

FIGURE 1. Sketch of the idealized jet flow.

By matching the perturbed pressure and displacement at the jet boundaries one obtains the following relations originally given by Miles (1958):
For symmetric disturbances:

$$\coth\left[i\alpha H \beta_2\right] = \frac{\rho_1}{\rho_2}\frac{c_1^2}{(c_2-M)^2}\frac{\beta_2}{\beta_1}, \tag{4}$$

and for antisymmetric disturbances:

$$\tanh\left[i\alpha H\beta_2\right] = \frac{\rho_1}{\rho_2}\frac{c_1^2}{(c_2-M)^2}\frac{\beta_2}{\beta_1}. \tag{5}$$

ρ_j is the fluid density and H is the jet half width.

Before discussing the solutions of the above equations, it will prove useful to review the solutions for the case of a single vortex sheet separating two uniform semi-infinite regions.

By performing a similar linearized analysis and matching boundary conditions on either side of the single vortex sheet, one obtains

$$\rho_2^2(c_2-M)^4\beta_1^2-\rho_1^2c_1^4\beta_2^2 = 0. \tag{6}$$

Miles (1958) solved the initial-value problem for the vortex sheet to determine which of the solutions of (6) were physically relevant and he found that in the range

$$|U_2-U_1| < (a_1+a_2), \tag{7}$$

both stable and unstable modes exist which are specified by the wave speed c,

$$c = \tfrac{1}{2}M \pm [(M^2+1)^{\frac{1}{2}}-\tfrac{1}{4}(M^2+4)]^{\frac{1}{2}}\, i, \tag{8}$$

for the special case $a_1 = a_2$.

When

$$a_1+a_2 < |U_2-U_1| < (a_1^{\frac{2}{3}}+a_2^{\frac{2}{3}})^{\frac{3}{2}}, \tag{9}$$

the previous solutions are valid, but one finds in addition one neutral solution which in the case of equal speeds of sound is given by

$$c = \tfrac{1}{2}M, \tag{10}$$

corresponding to the range

$$2 < M < 2\sqrt{2}, \tag{11}$$

as given by (9).

Finally, when

$$(a_1^{\frac{2}{3}}+a_2^{\frac{2}{3}})^{\frac{3}{2}} < |U_2-U_1|, \tag{12}$$

the unstable and damped solutions disappear entirely and one is left with three distinct neutral modes. For equal speeds of sound these solutions are given by (8) and (10) with the second term in (8) now being real.

Although (12) gives a criterion for the absence of unstable two-dimensional modes, Fejer & Miles (1963) have also shown that the system is always susceptible to three-dimensional instabilities, no matter how large the velocity jump, by considering disturbances which propagate at a sufficiently large angle to the x axis in the x, z plane. We will consider two-dimensional disturbances with the knowledge that the final results can then be modified to incorporate three-dimensional effects.

We note that Miles's neutral solutions, found in the velocity ranges of (9) and (12) will playan important part in high-speed noise production since the forcing of these solutions by acoustic radiation may lead to a resonance condition in which the magnitudes of the shear layer's reflexion and transmission coefficients become infinite. Miles (1956) gives the reflexion coefficient, R, for a plane wave in region (2) as

$$R = \left[1-\frac{\rho_2}{\rho_1}\frac{\beta_1}{\beta_2}\frac{(c_2-M_2)^2}{(c_1-M_1)^2}\right]\Big/\left[1+\frac{\rho_2}{\rho_1}\frac{\beta_1}{\beta_2}\frac{(c_2-M_2)^2}{(c_1-M_1)^2}\right]. \tag{13}$$

This result is valid for arbitrary velocities. Miles's paper should be consulted for the proper choice of signs of β_j in this case of purely time periodic waves.

Returning to the jet we look for solutions of (4) and (5) and ask if there is some frequency or wavelength which results in a definite maximum growth rate of the jet's instabilities.

Two types of instability often studied are, (i) the temporal instability having

$$\operatorname{Im}\alpha = 0, \tag{14}$$

so that there is growth in time but a sinusoidal variation in space, and (ii) the spatial instability having

$$\operatorname{Im}(\alpha\bar{c}) = \operatorname{Re}\alpha \operatorname{Im}\bar{c} + \operatorname{Im}\alpha \operatorname{Re}\bar{c} = 0, \tag{15}$$

so that there is growth in space but a sinusoidal variation in time at any point. Many disturbances in a jet appear to be of the latter type, i.e. a fixed pattern of oscillations relative to the jet nozzle, but other disturbances may also have a time-dependent character if they are triggered off at either random or periodic times.

It will now be shown that there is no absolute maximum growth rate corresponding to some particular wavelength, but that the growth rate in either time or space increases without limit for increasingly large wave-number.

In the case of temporal instabilities in the incompressible Rayleigh jet problem one can easily show from (4) that for equal densities

$$\left.\begin{aligned} \operatorname{Im}\bar{c} &= \pm\frac{U\coth^{\frac{1}{2}}(\alpha H)}{1+\coth(\alpha H)}, \\ \operatorname{Re}\bar{c} &= \frac{U\coth(\alpha H)}{1+\coth(\alpha H)}, \end{aligned}\right\} \tag{16}$$

for symmetric modes and a similar expression for antisymmetric modes. Thus $\operatorname{Im}\omega = \alpha\operatorname{Im}\bar{c}\to\infty$ as $\alpha\to\infty$ with $\operatorname{Im}\bar{c}$ and $\operatorname{Re}\bar{c}$ tending to definite limits. Let us then look for solutions in the general case in the limit $|\alpha|\to\infty$ with $\bar{c}$ again approaching a limit.

With $\bar{c}$ complex, $i\alpha\beta_2$ will generally be complex so that if $|\alpha|\to\infty$,

$$\coth(i\alpha\beta_2 H)\to\pm 1,$$

the choice of sign being due to the freedom allowed in choosing the sign of β_2. In this limit both (4) and (5) reduce to (6), the equation governing the single shear layer with the corresponding values of $\bar{c}$. The reason for this is that whether we have spatial or temporal jet instabilities, the disturbance given by (1) decays away from the shear layer with an arbitrarily large exponential rate for an arbitrarily large value of $|\alpha|$ so that the effects of one of the jet boundaries is not felt by the second one. To show that this solution is indeed approached as $|\alpha|\to\infty$ one perturbs (4) and (5) by letting

$$\bar{c} = \bar{c}_0 + \tilde{c}, \tag{17}$$

where $\bar{c}_0$ is a solution of the infinite $|\alpha|$ jet, or single vortex sheet case, and one finds that

$$\left.\begin{aligned}\tilde{c} &= \pm 2e^{-2\bar{x}}[\cos 2\bar{y} - i\sin 2\bar{y}]\Big/\left[\frac{\bar{c}_0 - U}{(\bar{c}_0 - U)^2 - a_2^2} - \frac{\bar{c}_0}{\bar{c}_0^2 - a_1^2} - \frac{2U}{\bar{c}_0(\bar{c}_0 - U)}\right],\\ \bar{x} &= -[\operatorname{Im}\alpha \operatorname{Re}\beta_{20} + \operatorname{Re}\alpha \operatorname{Im}\beta_{20}]H,\\ \bar{y} &= [\operatorname{Re}\alpha \operatorname{Re}\beta_{20} - \operatorname{Im}\alpha \operatorname{Im}\beta_{20}]H,\end{aligned}\right\} \tag{18}$$

where the plus sign applies to symmetric disturbances and the minus to antisymmetric disturbances by arbitrarily choosing $\operatorname{Im}\beta_2 > 0$. In the case of temporal instabilities

$$\operatorname{Im}\omega = \alpha \operatorname{Im}\bar{c} \tag{19}$$

and in the case of spatial instabilities

$$\operatorname{Im}\alpha = -\operatorname{Re}\alpha \operatorname{Im}\bar{c}/\operatorname{Re}\bar{c}. \tag{20}$$

Thus in either case (14) and (15) show that as $|\alpha| \to \infty$, $\bar{x} \to \infty$, $\tilde{c} \to 0$, and the growth rates given by (19) and (20) become linear in $\operatorname{Re}\alpha$ except for the high Mach number range to be discussed next.

Although the single vortex sheet is stable for all two-dimensional disturbances when the velocity jump is large enough to satisfy (12), the jet is always unstable no matter how high the Mach number. The analysis for this latter case proceeds by writing (4) and (5) in the form

$$i\alpha\beta_2 H = -\tfrac{1}{2}\ln[\pm R], \tag{21}$$

with the plus corresponding to (4) and the minus to (5). R is as written in (13), however, whereas $\bar{c}$ was real in the case of waves incident upon a vortex sheet, both $\bar{c}$ and R will be complex in the case of a jet. The results to be obtained for the jet are independent of the sign convention used for β_2 in R.

Breaking (21) into its real and imaginary parts the equations governing symmetric disturbances become

$$[\operatorname{Re}\alpha \operatorname{Im}\beta_2 + \operatorname{Im}\alpha \operatorname{Re}\beta_2]H = \tfrac{1}{2}\ln|R|, \tag{22}$$

$$[\operatorname{Re}\alpha \operatorname{Re}\beta_2 - \operatorname{Im}\alpha \operatorname{Im}\beta_2]H = -\tfrac{1}{2}\arg(R) \pm n\pi \quad (n = 0, 1, 2, \ldots), \tag{23}$$

with R in (23) replaced by $-R$ for the antisymmetric case. By using either (14) or (15) for temporal or spatial instabilities respectively, (22) and (23) become two equations in the two unknowns $\operatorname{Im}\bar{c}$, $\operatorname{Re}\alpha$ as functions of $\operatorname{Re}\bar{c}$.

Let us consider, for example, temporal instabilities in the large α limit where in the high velocity range given by (12) we expect $\operatorname{Im}\bar{c}$ to be small since the infinite α solution is the neutral vortex sheet solution having $\operatorname{Im}\bar{c} = 0$. If $\operatorname{Im} c_2 \ll 1$ one may make the following approximation:

$$\operatorname{Im}\beta_2 \simeq \frac{-(\operatorname{Re} c_2 - M)\operatorname{Im} c_2}{[(\operatorname{Re} c_2 - M)^2 - 1]^{\frac{1}{2}}}, \tag{24}$$

$$\operatorname{Re}\beta_2 \simeq -[(\operatorname{Re} c_2 - M)^2 - 1]^{\frac{1}{2}}, \tag{25}$$

with the arbitrary choice of the minus sign before β_2. Then (22) and (23) become

$$\operatorname{Im} c_2 \simeq \frac{-[(\operatorname{Re} c_2 - M)^2 - 1]^{\frac{1}{2}}}{2\alpha H(\operatorname{Re} c_2 - \mathrm{M})}\ln|R|, \tag{26}$$

$$\alpha H[(\operatorname{Re} c_2 - M)^2 - 1]^{\frac{1}{2}} \simeq \tfrac{1}{2}\arg R \pm n\pi \approx \pm n\pi. \tag{27}$$

The initial assumption of $\operatorname{Im} c_2 \ll 1$ follows from (26) if $\alpha H \gg 1$ which will happen according to (27) if n is sufficiently large. The infinite α solutions according to (8) and (10) have $(\operatorname{Re} c_2 - M) < 0$ and with our choice of β_2 one finds that $|R| > 1$ so that $\ln |R| > 0$ and hence $\operatorname{Im} c_2 > 0$, i.e. we have an instability. It is thus seen from (26) that $\operatorname{Im}\omega = O(a_2/H)$ except near the single vortex sheet resonant condition for which $R^{-1} \rightarrow 0$ (for the assumed sign of β_2 or $R \rightarrow 0$ if the other sign had been chosen) where $\operatorname{Im}\omega = O((a_2/H)\ln \alpha H)$ so that the growth rate depends logarithmically on α for large α.

This can be better understood by the following model. If one now moves to a reference frame translating to the right with the speed U_2, then the jet fluid will appear to be stationary and the external fluid will have a mean motion in the negative x direction. If $\operatorname{Im} c_2 \ll 1$ ray theory may be able to give us an adequate description. Consider that some disturbance has been excited at a point $x = 0$ within the now stationary jet fluid. The emitted rays will travel along straight line trajectories until they strike one of the vortex sheets whereupon they will be reflected at the incident angle, following the new straight line trajectory until they are reflected by the second vortex sheet and so on. After each reflexion the incident wave disturbance is multiplied by a factor of R corresponding to its angle of propagation. After many reflexions one could then write

$$\phi = \phi_0 R^{\sigma x}, \tag{28}$$

where σ equals the number of reflexions per unit length along the x axis, which in this case would be

$$\sigma = \frac{\tan\theta}{2H}, \tag{29}$$

where θ is the propagation angle measured with respect to the x axis, as is illustrated in figure 2.

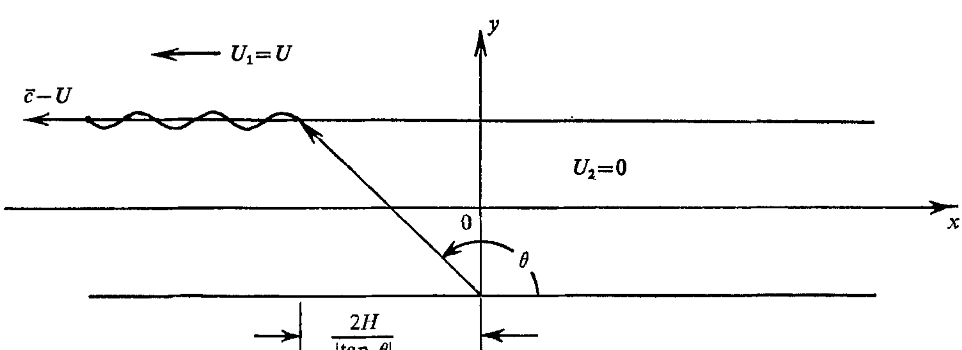

FIGURE 2. Sketch illustrating symbols used in discussion of jet resonance.

The magnitude of ϕ is $\quad |\phi| = |\phi_0|\,|R|^{x\tan\theta/2H}$ (30)

and its change along the x axis is

$$\frac{d|\phi|}{dx} = |\phi|\frac{\tan\theta}{2H}\ln|R|. \tag{31}$$

If one now follows an individual pulse travelling at the angle θ, then the time required for it to travel a unit distance (including reflexions) in the x direction is $\cos\theta$ times the speed of sound, so that changes in time following the pulse may be related to changes in x in the following manner

$$\frac{\partial}{\partial t} = a_2 \cos\theta \frac{\partial}{\partial x}, \tag{32}$$

with

$$\left.\begin{aligned} \cos\theta &= \frac{1}{c_2 - M}, \\ \tan\theta &= -[(c_2-M)^2-1]^{\frac{1}{2}}. \end{aligned}\right\} \tag{33}$$

Upon substitution of (32) and (33) into (31) one finds that the exponential growth rate that this model predicts is identical to that given by (26) in the limit of small $\operatorname{Im} c_2$. Note also that although the amplified rays propagate upstream through the jet fluid they actually grow in the downstream direction relative to the external region due to the large convective velocity of the jet.

The analysis follows analogously for spatial instabilities, and one finds, either by use of the model described by (31) with the appropriate change of reference frame or from (20), (22)–(25), that the spatial growth rate is

$$\operatorname{Im}\alpha \simeq \frac{[(\operatorname{Re}c_2 - M)^2 - 1]^{\frac{1}{2}} \ln|R|}{2H[M(\operatorname{Re}c_2 - M)+1]}. \tag{34}$$

3. Conclusions

The double vortex sheet model shows that the growth rates of instabilities are unbounded as α increases. In reality the vortex sheets are actually shear layers of finite thickness. We first point out that the discontinuous velocity jump is a valid limit if we let the ratio of the shear layer thickness δ to wavelength λ tend towards zero. Graham & Graham have studied the case of a single shear layer having a linear velocity profile, and one can show from their analysis that the finite thickness profile solution does approach Miles's result for $\delta/\lambda \to 0$.

The effect of finite shear layer thickness on the jet may be inferred from the Rayleigh jet case. Rayleigh studied a trapezoidal velocity profile and showed that the rectangular profile result is attained in the limit $\delta/\lambda \to 0$. However, of greater importance is the case when $\delta/\lambda = O(1)$, for then it is found that $\alpha \operatorname{Im}\bar{c}$ does reach a maximum. An analysis for the compressible jet is more difficult, but we would expect to find that the maximum value of the growth rate occurs when $\alpha\delta = O(1)$.

Current jets which experience screech generally operate in the lowest velocity range given by (7) or possibly in the lower portion of the region given by (9). In these regions there are fast growing modes which behave qualitatively very much like those of the incompressible jet. In general it appears that the jet is extremely susceptible to a wide range of instabilities. It constitutes a high gain broad band amplifier.

If the shear layer is thin, then frequencies of $O(\alpha/\delta)$ will result in the maximum rate of growth which will have its greatest initial effect near the shear layers with an exponential decrease in amplitude as the centre of the jet is approached.

According to the linearized theory the instability will spread rapidly into the core of the jet in a distance of $O(H)$. The disturbance will quickly grow to such a magnitude near the jet boundaries that it will develop into turbulence, possibly before the linear disturbance has a chance to penetrate into the bulk of the jet. A lower frequency instability, e.g. $O(a/H)$ (note that screech frequencies are of this order also), will result in slower growth rates, though the disturbance will still become large in a distance of $O(H)$, but a larger volume of the jet will be affected.

To summarize, the double vortex sheet model of the jet shows that compressible jets are basically as unstable as incompressible ones. Although the nature of the instability is different at very high jet speeds, the jet is always unstable no matter how high the jet Mach number.

In order to fully predict the behaviour of the jet, one must include the effects of non-linearity, finite shear layer thickness as well as other pertinent features of the real jet. It is likely though that modes can be excited which will greatly alter the characteristics of the jet.

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